MATHEMATICS IN PRE-SERVICE TEACHER EDUCATION AND THE QUALITY OF LEARNING: THE MONTY HALL PROBLEM

Fernando Luís Santos [(1)(3)] and António Domingos [(2)(3)]

(1) Piaget Institute, School of Education, Almada, Portugal ; fernando.santos@almada.ipiaget.pt

(2) Universidade Nova de Lisboa, Faculdade de Ciências e Tecnologia, Portugal; amdd@fct.unl.pt

(3) Research Centre for Education and Development (UIED), Portugal

If students acquire a new mathematical notion, according to Gray and Tall (1994), they pass through a proceptual divide. At a higher education institution in Portugal, students from different courses (education and business) came into contact with the Monty Hall problem in Statistics class. As a part of the learning environment the students have at their disposal all the technological apparatus they could use. The correct outcomes of two students (from the two courses) are analysed against the background of a model of analysis based on Tall's theory of the advanced mathematical thinking linked with SOLO taxonomy by Biggs and Collis (1982) and supported by Engeström (2001) third generation model of Activity Theory. In particular, the two different outcomes show that students can attain the same level, but may be operating in different levels: procedural thinking and proceptual thinking.

Keywords: Advanced mathematical thinking, proceptual divide, probability, quality of learning, SOLO taxonomy.

INTRODUCTION

If we look at a mathematical object, we can think of it relating with its mathematical definition, properties, relations, processes and procedures. Gray and Tall (1994) uses the dichotomy between procedure and concept to characterize the proceptual divide. On the one hand procedure relates to routine manipulations, focused on performance and it's somehow inflexible, on the other hand conceptual knowledge calls for relationships and flexible concepts.

> In this sense we can talk about the procept 6. It includes the process of counting 6, and a collection of other representations such as 3+3, 4+2, 2+4, 2 x 3, 8–2, etc. All of these symbols are considered by the child to represent the same object, though obtained through different processes. But it can be decomposed and recomposed in a flexible manner. (Gray & Tall, 1994, p. 6-7)

The above definition reflects the cognitive reality by using the term *procept* to translate the flexibility of the notion starting from an "*elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object." (Gray & Tall, 1994, p.6)

This paper is a report of one episode of an ongoing study of this topics and the underlying question is: does the analysis proposed for student outcomes access the complexity of their thought, and furthermore can it reason about the quality of their mathematical learning? To approach an answer we present an analysis of two correct outcomes, but evaluated in different levels of mathematical thought. Technologies used in this paper are seen as tools to learn mathematics rather than the mathematics used as an excuse to use technological skills.

THEORETICAL FRAMEWORK

Advanced mathematical thinking

In 1988, Tall argued that *advanced mathematical thinking* could be seen in two different ways: (I) Thought related to advanced mathematics, or (ii) advanced ways of mathematical thought (Tall, 1988).

But, what is *advanced mathematical thinking?* Since Ervynck coined the expression in 1985 there is a discussion about it, to some it relates to cognitive changes between secondary and higher education students, others stand for the origin of the cognitive conflicts inherent to mathematical thought. To Tall (1988) advanced mathematical thinking is any part of problem-solving which includes the development of conceptual fields by abstraction.

Gray and Tall (1994) uses the *encapsulation* notion of a process in a mental object, rooted in the works of Piaget to support the cycles of assimilation and accommodation. The use of symbols brings itself and ambiguity between procedure and concept that they can define as a procept. The way students address this ambiguity seems to be the key for the quality of the mathematical learning.

Supported by the use of procepts the characteristics that makes a difference between two forms of thinking are: (i) procedural thinking focused on procedure, mathematical objects are concrete entities that can be manipulated based on some rules; (ii) proceptual thinking focused on the flexibility and the ability to use a mathematical object in many ways. “This lack of a developing proceptual structure becomes a major tragedy for the less able which we call the proceptual divide.” (Gray & Tall, 1994, p. 18)

SOLO (Structure of the observed learning outcomes) taxonomy

The emphasis on the quality of student outcomes is a key point for the use of this taxonomy in the analytical model proposed. Its focus is not on the correctness of the outcomes, but in their nature, coded in SOLO levels.

To Biggs and Collis (1982) the quality of learning of a student depends on external stimulations, such as the quality of teaching and internal stimulations like the development stage, its previous knowledge about the subject and motivation. But it is hard to identify this quality solely considering a development stage, if we change focus to their outcomes we can identify patterns. These patterns are important components for the terminology used in the taxonomy.

We describe the basic features of the SOLO taxonomy, adapted from Biggs and Collis (1982):

1. *Pre-structural*, the outcomes provide non related information, loose and disorganized with minimal capacity;
2. *Uni-structural*, the outcomes provide simple connections, does not identify its importance, jumps to conclusions on a single aspect;
3. *Multi-structural*, the outcomes provide some connections but without a unifying vision, can isolate relevant data, work with algorithms and perform simple procedures;
4. *Relational*, the outcomes make complex connections, use relevant data and interrelations, explaining the causes;
5. *Extended abstract*, the outcomes goes beyond the topic, make generalizations, use relevant data with no need to give closed responses.

Activity Theory

Initially developed by Vygotsky and Leont'ev centred in the triangle of activity guided by objects (first generation), Engeström (2001) expanded the original centred now in activity (as a process) reflecting the actions and interactions of the subject with the context within learning occurs (second generation). With the notion of activity network a third generation emerged with the centre now in, at least, two activity systems in interaction.

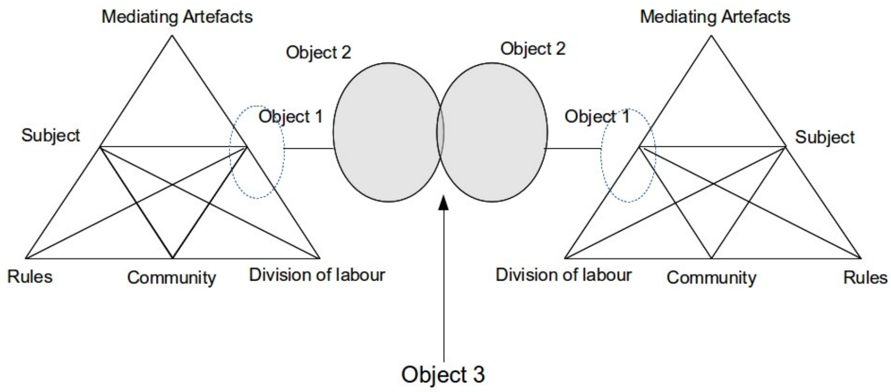

Figure 1. Two activity systems in interaction (Engeström, 2001).

In figure 1, the object goes from an initial outcome (object 1; seen as the answer made by subject and/or the solution by the teacher) to a meaningful outcome constructed by their activity systems (object 2; seen as the expected outcome by each student and teacher) leading the intersection of both expected outcomes to produce a shared object and their evaluation related to the quality of student learning (object 3; seen as a constructed understanding on the outcomes of both activity systems). This is not a static situation, therefore it shows only a picture of a specific outcome.

This third generation of activity theory could be summarized by Engeström (2001) seeing the object of activity as a moving target for an expansive transformation in activity systems supported by the contradictions as a source of development. These contradictions are not conflicts since it evolves a dialectic and multi-directional relation supported by Marx and Hegel in the contradictions of the dialectic relation.

THE MONTY HALL PROBLEM

> In a TV contest, a contestant chooses one of three doors; behind one of the doors there is a prize and behind the other two there is nothing. After the competitor choose a door, the host opens one of the other and reveals that there is no prize. The host then asks the competitor's choice whether to keep or want to switch. It is advantageous, in statistical terms, to switch or keep?

The solution to this problem caused a great deal of controversy among mathematicians since 1990 answer by Marilyn vos Savant that the contestant should switch. The original problem is based on the TV show *Let's make a deal* starring Monty Hall and it's been discussed since 1975 (at least).

METHODOLOGICAL APPROACH

These episodes are taken from one larger ongoing study, this specific episode was designed based on discussion classes from an education and a business courses in which two students (let's call them Raquel and Mariana) presented different solutions both correct, using all the technological apparatus at their disposal (smartphones, tablets, internet access, and so on). In these episodes one of us acted as a teacher and as a researcher and a two classes are reported, totalling four hours of work, both classes about the Monty Hall problem.

One of the goals of this kind of class (discussion) was to enhance not just the resolution of common exercises in statistics and probability, but to create a kind of problem based learning sustained by a community of enquiry. In these two classes, a variety of mathematical problems are stated and students discussed possible solutions for a 10 or 15 minutes time, then they have 20 minutes to write solutions in order to present to their classmates in the remaining time.

The outcomes were analysed based on SOLO levels and their attributes and deepened by Tall theories covering aspects of procepts and proceptual divide, supported by the third generation of activity theory scheme.

Raquel, a second year student of Business had some interest about the problem and decided to work alone. Grabbed her tablet and searched (in Portuguese) for similar problems switching to English when she found some articles related to Monty Hall.

Her solution is based on the conditional probability, namely *Bayes Theorem* using a decision tree as shown in figure 2 bellow:

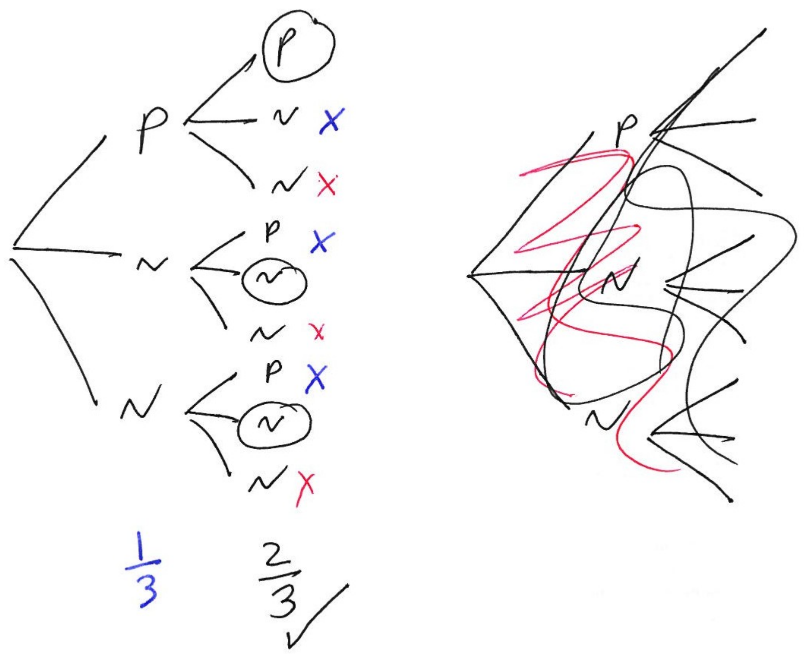

Figure 2. Sketch presented by Raquel to explain her solution where P stands for "Prize" ("Prémio" in Portuguese) and N stands for "Nothing" ("Nada" in Portuguese).

In this decision tree Raquel explains that the red crosses stand for the time that the host reveal one door. Then she continues her explanation (the dialogues were held in Portuguese):

Raquel: The blue crosses are when we stay with the same door leading to Prize-Nothing-Nothing or one third probability.

Teacher: …

Raquel: The black circles are when we change doors leading to Nothing-Prize-Prize or two thirds probability, so we must change to get a better chance to win the prize.

Teacher: Isn't this sketch to confuse...we just understand it when you explain...

Raquel then turns to her tablet and five minutes later comes up with this solution (figure 3):

$$P(A/O) = \frac{P(A) \cdot P(O/A)}{P(O)}$$

$$P(C/O) = \frac{P(C) \cdot P(O|C)}{P(O)}$$

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

$$P(O/A) = \frac{1}{2} \; ; \; P(O/B) = 0 \; ; \; P(O|C) = 1$$

$$P(O) = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0 + \frac{1}{3} \times 1$$

$$= \frac{1}{2}$$

$$P(A/O) = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{1}{3}$$

$$P(C/O) = \frac{\frac{1}{3} \times 1}{\frac{1}{2}}$$

$$= \frac{2}{3}$$

$$P(A/O) \neq P(C/O)$$

Figure 3. Calculations made by Raquel using Bayes Theorem.

And explained that A, B and C are the events, O is the event that the host opens door number 2 so by calculations made with Bayes Theorem the result is the same of the sketch.

At this time we realize that for her, the problem is solved. When we analyse this episode, Raquel tried, with some success, to use a decision tree to explain the solution of the problem, but the drawing was too confusing, we evaluated this attempt as *relational* in SOLO taxonomy because she makes some complex connections, explains her steps and analyses the solution, but when she was questioned about the confusing design she gave other kind of response, a more *mathematical* solution with the help of the conditional probability.

Her outcome is now classified as *multi-structural* level because she just worked with the algorithms. She just found a webpage with the solution and just copied to the paper. Somehow she felt frustrated that she can't draw a better example and by the use of technology took refuge on the more familiar calculations and algorithms.

The analysis of Raquel outcomes evidenced a procedural thinking, even more when she was asked to explain the first outcome and she goes back to the algorithms evidencing some contradictions in her activity system namely in the *rules* and in the *mediating artefacts* that produces her outcome, as we can see in figure 4.

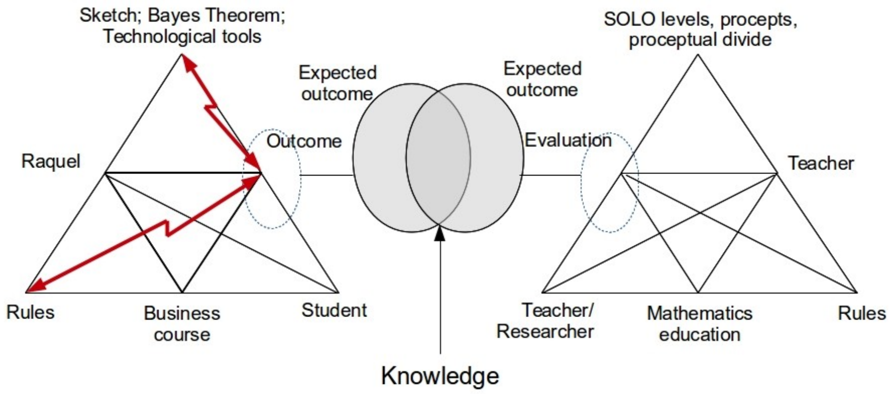

Figure 4. Activity system drawn from Raquel outcomes.

This activity system also evidences a difference on the expected outcome from the two isolated activity systems (student and teacher), the intersection made from this kind of Venn diagram in the middle shows a third object that emerges from the connection of both activity systems.

The next episode with Mariana went up differently. Mariana is a third year student in Education, with no problems using technology, so she chooses the Monty Hall problem and tried to simulate the solution using a spreadsheet. The next section describes all the steps Mariana made to build the simulation:

> On the first cell she wrote =INT(RAND()*3)+1 to generate a whole number from 1 to 3, on the second cell used the same formula to generate a new random number (from 1 to 3) to indicate the door that the contestant could choose, for the third cell the formula was more complicated, but with the help of some spreadsheet cheat sheets she got a conditional formula:
>
> =IF(C4=B4;IF(B4=1;IF(RAND()<0,5;2;3);IF(B4=3;IF(RAND()<0,5;1;2);IF(RAND()< 0,5;1;3)));IF(C4=1;IF(B4=2;3;2);IF(C4=2;IF(B4=1;3;1);IF(B4=2;1;2))))
>
> This formula generates one of three numbers avoiding the numbers of the first two cells, it is the door that the host opens. In the fourth cell she wrote:
>
> =IF(D4=1;IF(C4=2;3;2);IF(D4=2;IF(C4=1;3;1);IF(C4=1;2;1)))
>
> Other conditional formula to prevent the random number to be the one in cell three or in cell one. Now the next formula served to check if the contestant win or lose: number 1 if cell 1 and for match, 0 if it doesn't match:
>
> =IF(E4=B4;1;0).
>
> To finalize, after she copied the first line 100 times she just made a sum from this hundred counts on the next cell:
>
> =SUM(F4:F1003)

And made a percentage from the value. As the number were randomized she got values around 67,5% every time arriving to the conclusion that it is advantageous to switch.

This method to find the result is a convincing demonstration and could be found in many web pages around internet, but in this case Mariana didn't just copy the formulas or the demonstration spreadsheets that can be downloaded, she explained to her classmates and replicated the simulation.

Although this simulation isn't a traditional mathematical proof it shows that technological tools could be used to give a new look to mathematics, this outcome was classified as *relational* in SOLO levels close related to the *extended abstract* because, on the one hand Mariana makes complex relations, explain the causes, integrates several areas of knowledge, on the other hand she goes beyond the topic making generalizations to other concepts.

The activity system is different from the one presented in figure 4 although the system contradictions are the same in figure 5.

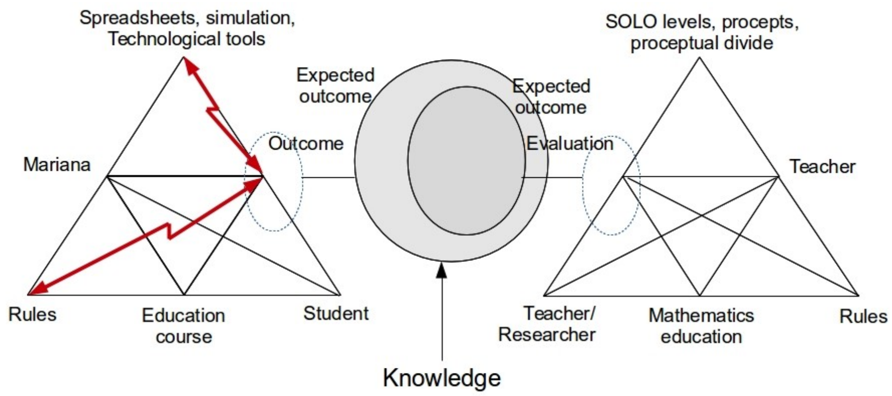

Figure 5. Activity system drawn from Mariana outcome.

This activity system shows the use of more complex procepts clearly a sign of proceptual thinking due to the use of simulation that evidences a proceptual divide. In this case the third object clearly surpasses the expected outcomes of the teacher.

FINAL REMARKS

Both these students worked alone, and both used several technological tools at their disposal, but, possibly due to their different areas and backgrounds their outcomes exposes a proceptual divide. Raquel used a procedural type of thinking, with elementary procepts and started with a *relational* level on the SOLO taxonomy and ended with a *multi-structural* level, we might think as a regression, but one of the characteristics of procedural thinking is the refuge on algorithms and procedures well known without space for new knowledge that she started but was unable to process.

Mariana on the other hand, even with the aid of some simulations found on Internet could reproduce and explain all the processes evolved in her outcome, evidencing a proceptual thinking with some meaningful combination of elementary procepts to form a procept with a flexible combination of derived facts surpassing the barrier of proceptual divide.

Although the ongoing study reported in this paper is not yet closed, it already conjectures interesting results, not only on the model of analysis used to evaluate the outcomes but also on the evidences showing an emergent curriculum for pre-service teacher education (in this institution) based on the outcomes of this study.

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